Shape design for superconductors governed by H(curl)-elliptic variational inequalities

Antoine Laurain

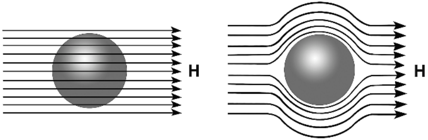
Instituto de Matemática e Estatística, Universidade de São Paulo.

joint work with: Malte Winckler and Irwin Yousept

Dynamics, Equations and Applications 2019

Type-II superconductivity

- Superconductivity: if a superconductor is cooled down below its critical temperature T_c, then it loses its electrical resistivity.
- Meissner effect: If an external weak magnetic field (below a certain critical level H_c) is applied to a superconductor at a temperature below its critical temperature T_c, then the magnetic flux is completely expelled from the superconductor.



- Applications: magnetic shielding, magnetic resonance imaging (MRI), magnetic confinement fusion technologies, high-energy particle accelerators, magnetic levitation technologies, magnetic energy storage.
- Type-II superconductors admit higher critical temperatures (high-temperature superconductivity) and greater critical values of magnetic field than type-I superconductors.

Bean critical-state model for HTS

- A critical-state model describing the magnetization process of penetration and exit of magnetic flux in type-II (high-temperature) superconductors was proposed in [Bean 1962,1964].
- Bean's model describes a nonlinear and non-smooth constitutive relation between the (total) current density and the electric field:

  (B1) the current density strength $|\boldsymbol{J}|$ cannot exceed the critical current j_c,
  (B2) if $|\boldsymbol{J}|$ is strictly less than j_c, then the electric field $\boldsymbol{E}$ vanishes,
  (B3) the electric field $\boldsymbol{E}$ is parallel to $\boldsymbol{J}$.

- In the recent past, the Bean critical-state model for HTS (high-temperature superconductivity) has been extensively studied by several authors.
- In the full 3D Maxwell case it gives rise to a hyperbolic Maxwell variational inequality (VI) of the second kind [Yousept 2017,2019].

$\boldsymbol{H}(\mathbf{curl})$-elliptic variational inequality

- Hyperbolic Maxwell VI of the second kind, full 3D Maxwell case:

$$\int_\Omega \epsilon \frac{d}{dt}\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) - \boldsymbol{H}(t)\cdot\mathbf{curl}(\boldsymbol{v}-\boldsymbol{E}(t))\,dx + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{E}(t))$$
$$\geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,dx \quad \text{for a.e. } t\in(0,T) \text{ and all } \boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl}),$$
$$\mu\frac{d}{dt}\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = 0 \text{ for a.e. } t\in(0,T)$$
$$(\boldsymbol{E}(0),\boldsymbol{H}(0)) = (\boldsymbol{E}_0,\boldsymbol{H}_0)$$

- Employing an implicit Euler time discretization leads to an elliptic VI [Winckler, Yousept 2019]:

$$\int_\Omega \epsilon\boldsymbol{E}\cdot(\boldsymbol{v}-\boldsymbol{E}) + \nu\,\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}(\boldsymbol{v}-\boldsymbol{E})\,dx + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{E})$$
$$\geq \int_\Omega \boldsymbol{F}\cdot(\boldsymbol{v}-\boldsymbol{E})\,dx \quad \text{for all } \boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl})$$

$\boldsymbol{H}(\mathbf{curl})$-elliptic variational inequality

- $\Omega \subset \mathbb{R}^3$ is bounded Lipschitz, $B \subset \Omega$ and $\omega \in \mathcal{O}$ is an admissible superconductor shape, with

$$\mathcal{O} := \{\omega \subset B \ : \ \omega \text{ is open and } L\text{-Lipschitz}\}.$$

- $\boldsymbol{E} = \boldsymbol{E}(\omega) \in \boldsymbol{H}_0(\mathbf{curl})$ electric field solution of

$$a(\boldsymbol{E}, \boldsymbol{v} - \boldsymbol{E}) + \varphi_\omega(\boldsymbol{v}) - \varphi_\omega(\boldsymbol{E}) \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{E})\, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}),$$

- non-smooth L^1-type functional

$$\varphi_\omega \colon \boldsymbol{L}^1(\Omega) \to \mathbb{R}, \quad \boldsymbol{v} \mapsto j_c \int_\omega |\boldsymbol{v}(x)|\, dx.$$

- elliptic **curl-curl** bilinear form $a \colon \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}_0(\mathbf{curl}) \to \mathbb{R}$

$$a(\boldsymbol{v}, \boldsymbol{w}) := \int_\Omega \nu\, \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{w}\, dx + \int_\Omega \varepsilon \boldsymbol{v} \cdot \boldsymbol{w}\, dx,$$

- $j_c > 0$ is the critical current density of the superconductor ω, $\epsilon, \nu \colon \Omega \to \mathbb{R}^{3\times 3}$ are the electric permittivity and the magnetic reluctivity, $\boldsymbol{f} : \Omega \to \mathbb{R}^3$ is the applied current source.

Optimal HTS design problem

- $\Omega \subset \mathbb{R}^3$ is bounded Lipschitz, $B \subset \Omega$ and $\omega \subset B$.
- In the numerical tests we take $\Omega = [-2, 3]^3$ and $B = [0, 1]^3$.
- We consider the following shape optimization problem

$$(P): \quad \min_{\omega \in \mathcal{O}} J(\omega) := \frac{1}{2} \int_B \kappa |\boldsymbol{E}(\omega) - \boldsymbol{E}_d|^2 \, dx + \int_\omega dx,$$

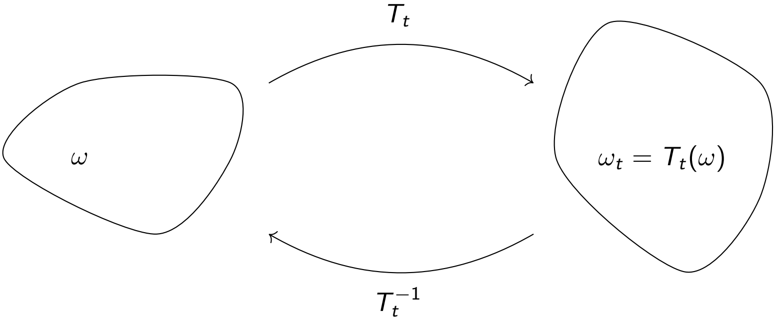
- $\boldsymbol{E}_d : B \to \mathbb{R}^3$ is a given target.
- $\kappa : B \to (0, \infty)$ is a weight coefficient .

Optimal design for variational inequalities

- Theoretical results:
  [Neittaanmäki, Sokołowski, Zolésio 1988], [Barbu, Friedman 1991], [Liu, Rubio 1991], [Denkowski, Migórski 1998], [Myśliński 2001], [Frémiot et al. 2009]
- Numerical results:
  [Kočvara, Outrata 1994], [Myśliński 2001], [Hintermüller, L. 2011], [Heinemann, Sturm 2016], [Führ, Schulz, Welker 2018].
- Topological derivative:
  [Sokołowski, Żochowski 2005], [Fulmański et al. 2007]

Domain perturbations and shape derivative

- $T_t : \Omega \to \Omega$ is a given diffeomorphism, with $\omega_t := T_t(\omega) \subset \Omega$.
- Example: $T_t(x) = (I + t\theta)(x)$ for $t \in [0, \tau]$ and given $\theta : \mathbb{R}^d \to \mathbb{R}^d$.
- Velocity $V := \partial_t T_t|_{t=0}$
- $\mathcal{O} \ni \omega \mapsto J(\omega) \in \mathbb{R}$ is a shape functional.
- Eulerian semiderivative $dJ(\omega)(V) := \lim_{t \searrow 0} \frac{J(\omega_t) - J(\omega)}{t}$
- $J(\omega)$ is shape differentiable if $V \mapsto dJ(\omega)(V)$ is linear and continuous.



$\boldsymbol{H}(\mathbf{curl})$-elliptic variational inequality

Assumptions

(A1) $B \subset \Omega$ is Lipschitz, $\boldsymbol{E}_d \in \boldsymbol{\mathcal{C}}^1(B)$, $\kappa \in \mathcal{C}^1(B)$, $j_c \in \mathbb{R}^+$.

(A2) $\epsilon, \nu \in L^\infty(\Omega, \mathbb{R}^{3\times 3}) \cap \mathcal{C}^1(B, \mathbb{R}^{3\times 3})$, symmetric and uniformly positive definite, i.e., there exist $\underline{\nu}, \underline{\epsilon} > 0$ such that

$$\xi^\mathsf{T} \nu(x) \xi \geq \underline{\nu} |\xi|^2 \text{ and } \xi^\mathsf{T} \epsilon(x) \xi \geq \underline{\epsilon} |\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^3.$$

(A3) $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega) \cap \boldsymbol{\mathcal{C}}^1(B)$.

- $a : \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}_0(\mathbf{curl}) \to \mathbb{R}$ is coercive and continuous.
- For every $\omega \subset \mathcal{O}$ the existence of a unique solution $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl})$ is covered by [Lions, Stampacchia 67]
- There exists a unique $\boldsymbol{\lambda} \in \boldsymbol{L}^\infty(\omega)$ such that

$$\begin{cases} a(\boldsymbol{E}, \boldsymbol{v}) + \int_\omega \boldsymbol{\lambda} \cdot \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ |\boldsymbol{\lambda}(x)| \leq j_c, \quad \boldsymbol{\lambda}(x) \cdot \boldsymbol{E}(x) = j_c |\boldsymbol{E}(x)| \text{ for a.e. } x \in \omega. \end{cases}$$

Penalized shape optimization approach

- Motivation: shape sensitivity analysis requires the differentiability of the mapping $\boldsymbol{E} \mapsto \boldsymbol{\lambda}$ in $\boldsymbol{L}^2(\Omega)$, which is not guaranteed in general.

$$(P_\gamma): \quad \min_{\omega \in \mathcal{O}} J_\gamma(\omega) := \frac{1}{2} \int_B \kappa |\boldsymbol{E}^\gamma(\omega) - \boldsymbol{E}_d|^2 + \int_\omega dx,$$

where $\boldsymbol{E}^\gamma := \boldsymbol{E}^\gamma(\omega) \in \boldsymbol{H}_0(\mathbf{curl})$ solves the penalized dual formulation

$$\begin{cases} a(\boldsymbol{E}^\gamma, \boldsymbol{v}) + \int_\omega \boldsymbol{\lambda}^\gamma \cdot \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \\ \boldsymbol{\lambda}^\gamma(x) = \dfrac{j_c \gamma \boldsymbol{E}^\gamma(x)}{\max_\gamma\{1, \gamma |\boldsymbol{E}^\gamma(x)|\}} & \text{for a.e. } x \in \omega. \end{cases}$$

- $\max_\gamma : \mathbb{R}^3 \to \mathbb{R}$ is the Moreau-Yosida type regularization of the max-function given by

$$\max_\gamma\{1, x\} := \begin{cases} x & \text{if } x - 1 \geq \dfrac{1}{2\gamma}, \\ 1 + \dfrac{\gamma}{2}\left(x - 1 + \dfrac{1}{2\gamma}\right)^2 & \text{if } |x-1| \leq \dfrac{1}{2\gamma}, \\ 1 & \text{if } x - 1 \leq -\dfrac{1}{2\gamma}. \end{cases}$$

Minimizers for the penalized functional

Theorem

Let $\gamma > 0$ be fixed. Then, (P_γ) admits an optimal shape $\omega_\star^\gamma \in \mathcal{O}$.

Sketch of proof: take a minimizing sequence ω_n^γ in $\mathcal{O}$. The fact that the sets ω_n^γ are uniformly L-Lipschitz provides a subsequence which converges to $\omega_\star^\gamma \in \mathcal{O}$ in the sense of characteristic functions.
Then show that $\boldsymbol{E}_n^\gamma \to \boldsymbol{E}_\star^\gamma$ in $\boldsymbol{H}_0(\mathbf{curl})$ which implies

$$J_\gamma(\omega_n^\gamma) \to= J_\gamma(\omega_\star^\gamma).$$

Key property: use that $\boldsymbol{\Lambda}_\gamma$ is monotone, i.e.,

$$(\boldsymbol{\Lambda}_\gamma(\boldsymbol{w}_1) - \boldsymbol{\Lambda}_\gamma(\boldsymbol{w}_2), \boldsymbol{w}_1 - \boldsymbol{w}_2)_{L^2(\Omega)} \geq 0 \quad \forall \boldsymbol{w}_1, \boldsymbol{w}_2 \in \boldsymbol{L}^2(\Omega),$$

where

$$\boldsymbol{\Lambda}_\gamma \colon \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega), \quad \boldsymbol{\Lambda}_\gamma(\boldsymbol{e}) := \frac{j_c \gamma \boldsymbol{e}}{\max_\gamma\{1, \gamma|\boldsymbol{e}|\}}.$$

Shape-Lagrangian

- Lagrangian $\mathcal{L} : \mathcal{O} \times \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}_0(\mathbf{curl}) \to \mathbb{R}$:

$$\mathcal{L}(\omega, \boldsymbol{e}, \boldsymbol{v}) := \frac{1}{2} \int_B \kappa |\boldsymbol{e} - \boldsymbol{E}_d|^2 \, dx + \int_\omega \, dx + a(\boldsymbol{e}, \boldsymbol{v}) + \int_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{e}) \cdot \boldsymbol{v} \, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx$$

  with

$$\boldsymbol{\Lambda}_\gamma : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega), \quad \boldsymbol{\Lambda}_\gamma(\boldsymbol{e}) := \frac{j_c \gamma \boldsymbol{e}}{\max_\gamma\{1, \gamma|\boldsymbol{e}|\}}$$

- covariant transformation:

$$\Psi_t : \boldsymbol{H}_0(\mathbf{curl}) \to \boldsymbol{H}_0(\mathbf{curl}), \qquad \Psi_t(\boldsymbol{e}) := (D\boldsymbol{T}_t^{-\mathsf{T}} \boldsymbol{e}) \circ \boldsymbol{T}_t^{-1}.$$

- $G(t, \boldsymbol{e}, \boldsymbol{v}) := \mathcal{L}(\omega_t, \Psi_t(\boldsymbol{e}), \Psi_t(\boldsymbol{v}))$
- Use averaged adjoint method [Sturm 2015], [L., Sturm 2016] to compute the shape derivative.

Distributed shape derivative

Theorem (Distributed shape derivative)

Let $\gamma > 0$, $\omega \in \mathcal{O}$ and $\boldsymbol{\theta} \in \mathcal{C}_c^{0,1}(\Omega)$ with a compact support in B. Then the distributed shape derivative of J_γ is

$$dJ_\gamma(\omega)(\boldsymbol{\theta}) = \partial_t G(0, \boldsymbol{E}^\gamma, \boldsymbol{P}^\gamma) = \int_B S_1^\gamma : D\boldsymbol{\theta} + \boldsymbol{S}_0^\gamma \cdot \boldsymbol{\theta}\, dx,$$

where $S_1^\gamma \in L^1(B, \mathbb{R}^{3\times 3})$ and $\boldsymbol{S}_0^\gamma \in \boldsymbol{L}^1(B)$ are given by

$$\begin{aligned}
S_1^\gamma = {} & \Big[\frac{\kappa}{2}|\boldsymbol{E}^\gamma - \boldsymbol{E}_d|^2 + \chi_\omega - \nu \,\mathbf{curl}\, \boldsymbol{E}^\gamma \cdot \mathbf{curl}\, \boldsymbol{P}^\gamma + \varepsilon \boldsymbol{E}^\gamma \cdot \boldsymbol{P}^\gamma + \chi_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma) \cdot \boldsymbol{P}^\gamma \\
& - \boldsymbol{f} \cdot \boldsymbol{P}^\gamma\Big] \boldsymbol{I}_3 - \kappa \boldsymbol{E}^\gamma \otimes (\boldsymbol{E}^\gamma - \boldsymbol{E}_d) + \nu \,\mathbf{curl}\, \boldsymbol{E}^\gamma \otimes \mathbf{curl}\, \boldsymbol{P}^\gamma \\
& + \nu^\mathsf{T} \,\mathbf{curl}\, \boldsymbol{P}^\gamma \otimes \mathbf{curl}\, \boldsymbol{E}^\gamma - \boldsymbol{P}^\gamma \otimes \varepsilon \boldsymbol{E}^\gamma - \boldsymbol{E}^\gamma \otimes \varepsilon^\mathsf{T} \boldsymbol{P}^\gamma + \boldsymbol{P}^\gamma \otimes \boldsymbol{f} \\
& - \chi_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma) \otimes \boldsymbol{P}^\gamma - \boldsymbol{E}^\gamma \otimes \boldsymbol{\psi}^\gamma(\boldsymbol{E}^\gamma) \boldsymbol{P}^\gamma, \\
\boldsymbol{S}_0^\gamma = {} & \frac{\nabla \kappa}{2}|\boldsymbol{E}^\gamma - \boldsymbol{E}_d|^2 - \kappa D\boldsymbol{E}_d^\mathsf{T}(\boldsymbol{E}^\gamma - \boldsymbol{E}_d) + (D\nu^\mathsf{T} \,\mathbf{curl}\, \boldsymbol{E}^\gamma)\,\mathbf{curl}\, \boldsymbol{P} \\
& + (D\epsilon^\mathsf{T} \boldsymbol{E}^\gamma) \boldsymbol{P}^\gamma - D\boldsymbol{f}^\mathsf{T} \boldsymbol{P}.
\end{aligned}$$

Stability analysis of the shape derivative

Theorem (Stability with respect to γ)

Let $\omega \in \mathcal{O}$, then the following stability estimate holds

$$|dJ_\gamma(\omega)(\boldsymbol{\theta})| \leq C\|\boldsymbol{\theta}\|_{\boldsymbol{\mathcal{C}}^{0,1}(B)} \quad \forall \boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}_c^{0,1}(\Omega), \operatorname{supp} \boldsymbol{\theta} \subset\subset B,$$

with a constant $C = C(j_c, \kappa, \epsilon, \nu, \boldsymbol{f}, \boldsymbol{E}_d, B, \omega)$ independent of γ.

Convergence of regularized shape optimization problem

Theorem

Let $\{\gamma_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ be such that $\gamma_n \to \infty$ as $n \to \infty$. Then, there exists a subsequence of $\{\gamma_n\}_{n\in\mathbb{N}}$, still denoted by $\{\gamma_n\}_{n\in\mathbb{N}}$, such that the sequence of solutions $\{\omega^{\gamma_n}\}_{n\in\mathbb{N}}$ of (P_{γ_n}) converges towards an optimal solution $\omega_\star \subset \mathcal{O}$ of (P) in the sense of Hausdorff and in the sense of characteristic functions.

Moreover, $\{(\boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n}), \boldsymbol{\lambda}^{\gamma_n}(\omega^{\gamma_n}))\}_{n\in\mathbb{N}}$ and $(\boldsymbol{E}(\omega_\star), \boldsymbol{\lambda}(\omega_\star))$ satisfy

$$\lim_{n\to\infty} \|\boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n}) - \boldsymbol{E}(\omega_\star)\|_{H(curl)} = 0,$$
$$\lim_{n\to\infty} \|\boldsymbol{\lambda}^{\gamma_n}(\omega^{\gamma_n}) - \boldsymbol{\lambda}(\omega_\star)\|_{H_0(curl)^*} = 0,$$

where $\boldsymbol{\lambda}^{\gamma_n}(\omega^{\gamma_n})$ (resp. $\boldsymbol{\lambda}(\omega_\star)$) is extended by zero in $\Omega \setminus \omega^{\gamma_n}$ (resp. in $\Omega \setminus \omega_\star$).

Numerical tests

- regularization parameter: $\gamma = 7 \cdot 10^4$
- domains: $\Omega = [-2,3]^3$ and $B = [0,1]^3$
- material parameters: $\epsilon = \nu = \boldsymbol{I}_3$ and

$$\boldsymbol{f}(x,y,z) = \begin{cases} \dfrac{R(0,\ -z+0.5,\ y-0.5)}{\sqrt{(y-0.5)^2+(z-0.5)^2}} & \text{for } (x,y,z) \in \Omega_p, \\ 0 & \text{for } (x,y,z) \notin \Omega_p, \end{cases}$$

- pipe coil $\Omega_p \subset \Omega$ defined by

$$\Omega_p := \left\{ |z-0.5| \leq 0.5 \text{ and } \sqrt{(x-0.5)^2+(y-0.5)^2} \in [1.2, 1.6] \right\}.$$

- forward problem computed using Newton method, with a finite element discretization based on the first family of Nédélec's edge elements at roughly 2.000.000 DoFs.
- codes written in PYTHON with the open-source finite-element computational software FEniCS.

Descent direction

- let $\boldsymbol{V}_h \subset \boldsymbol{H}^1(B) \cap \boldsymbol{\mathcal{C}}^{0,1}(\overline{B})$ be the space of piecewise linear and continuous finite elements on B.
- positive definite bilinear form $\mathcal{B} : \boldsymbol{V}_h \times \boldsymbol{V}_h \to \mathbb{R}$,
- find $\boldsymbol{\Theta} \in \boldsymbol{V}_h$ such that

$$\mathcal{B}(\boldsymbol{\Theta}, \boldsymbol{\xi}) = -dJ_\gamma(\omega)(\boldsymbol{\xi}) \text{ for all } \boldsymbol{\xi} \in \boldsymbol{V}_h.$$

- $\boldsymbol{\Theta} \neq 0$ is a descent direction since $dJ_\gamma(\omega)(\boldsymbol{\Theta}) = -\mathcal{B}(\boldsymbol{\Theta}, \boldsymbol{\Theta}) < 0$. We choose

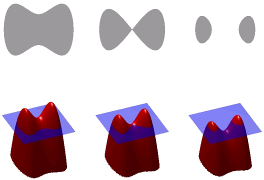
$$\mathcal{B}(\boldsymbol{\Theta}, \boldsymbol{\xi}) = \int_B \alpha_1 D\boldsymbol{\Theta} : D\boldsymbol{\xi} + \alpha_2 \boldsymbol{\Theta} \cdot \boldsymbol{\xi} \, dx + \alpha_3 \int_{\partial B} (\boldsymbol{\Theta} \cdot \boldsymbol{n})(\boldsymbol{\xi} \cdot \boldsymbol{n}) \, ds,$$

  with $\alpha_1 = 0.5$, $\alpha_2 = 0.5$ and $\alpha_3 = 1.0$.
- The geometry was optimized in the class of shapes with two symmetries with respect to the planes $x = 0.5$ and $y = 0.5$, using a symmetrized descent direction.

Level set method

The level set method [Osher, Sethian 1988] provides an implicit representation of ω_t via a level set function $\phi : \mathbb{R}^+ \times \Omega \to \mathbb{R}$.

$$\begin{aligned} \omega_t &:= \{x \in \omega_t \mid \phi(t,x) > 0\} \\ \omega_t^c &:= \{x \in \omega_t \mid \phi(t,x) < 0\} \\ \partial\omega_t &:= \{x \in \omega_t \mid \phi(t,x) = 0\} \end{aligned}$$



Given a descent direction $\boldsymbol{\Theta}$, the evolution of the level set function is determined by a Hamilton-Jacobi equation:

$$\partial_t \phi(t,x) + \boldsymbol{\Theta}(t,x) \cdot \nabla\phi(t,x) = 0 \text{ on } [0,\tau] \times \Omega.$$

First example

$\boldsymbol{E}_d \equiv 0$, $\kappa \equiv 8 \cdot 10^7$. Final functional value: 0.444, final volume: 0.278 which is only 27.8% of the volume of B. The E-field fraction in the cost functional amounts roughly to 0.166.

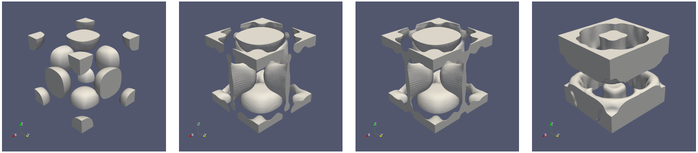


Figure: Shapes generated by the algorithm at iterations $0, 42, 45, 143$.

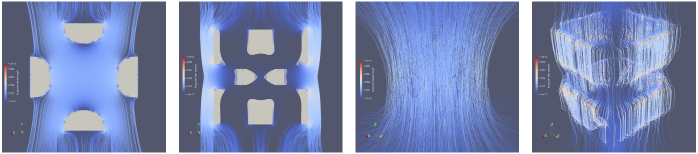


Figure: Different views on the magnetic field at the initial and the final iteration. a.)–b.): 2D slice in the center. c.)–d.): 3D view.

Second example

Here $\boldsymbol{E}_d$ is obtained by solving the VI with a superconducting ball ω_b with radius $r_b = 0.5$ inside B. Functional value: 0.223 (volume: 0.153 and electric field costs: 0.071). Optimal shape with around 70% less material.

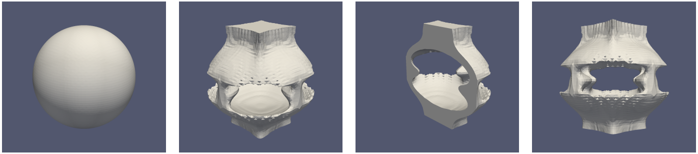


Figure: The original superconductor and the final shape generated by the algorithm. The third figure is the final shape clipped along the plane $x = 0.5$.

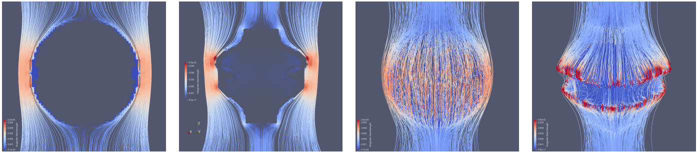


Figure: Different views on the magnetic field of the original and the final superconductor. Left: 2D slice in the center. Right: 3D view.

Convergence

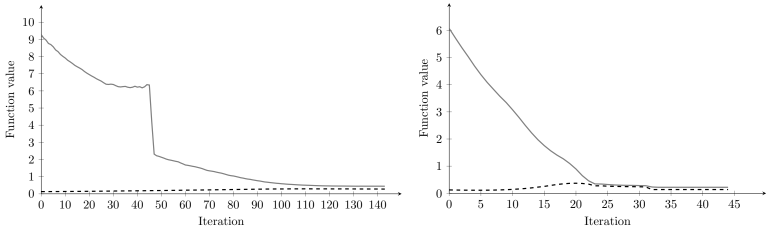


Figure: Function value (solid) and volume (dashed) during the algorithm. Left: First example. Right: Second example.

Conclusion

- Moreau-Yosida type regularization of the max-function of $\boldsymbol{H}(\mathbf{curl})$ elliptic VI.
- Distributed shape derivative for the regularized problem.
- Convergence of optimal shapes for the regularized problem and stability of the shape derivative with respect to regularization parameter γ.
- Numerical implementation using a level set method.
- The design optimization allows to save up to 70% of superconducting material.
- Paper submitted recently.

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THANK YOU!