Shape optimization of the ground state for two-phase conductors

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joint work with Carlos Conca and Rajesh Mahadevan

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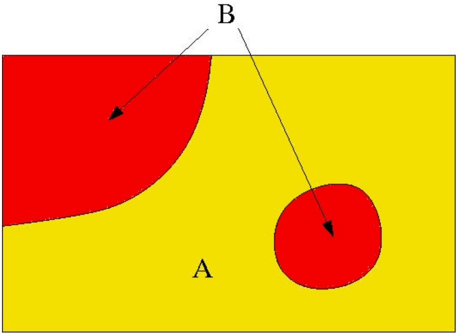
Problem Statement

Find the optimal distribution of two conducting materials A and B of given volume and conductivities α and β in a fixed domain Ω in order to minimize the ground state eigenvalue.



Problem setting

Eigenvalue problem

- $\Omega \subset \mathbb{R}^d, \quad 0 < \alpha < \beta, \quad 0 < m < |\Omega|$
- $B \subset \Omega$ measurable, $\quad A = \Omega \setminus B$
- $$\begin{aligned} -\mathrm{div}(\sigma(B)\nabla u) &= \lambda(B)u \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega\,. \end{aligned}$$
- $\sigma(B) = \alpha\chi_A + \beta\chi_B$, ($\chi_A$ and χ_B are indicator functions)
- $\lambda(B)$ is the first eigenvalue or ground state.

Shape Optimization Problem

$$\begin{aligned} &\text{minimize} && \lambda(B) \\ &\text{subject to} && B \in \mathcal{B} := \{B \subset \Omega, B \text{ measurable}, |B| = m\} \end{aligned}$$

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Known results

Existence

- Open question for general geometries of Ω.
- Existence of relaxed solutions:

-> Steven Cox and Robert Lipton (1996). "Extremal eigenvalue problems for two-phase conductors." In: *Arch. Rational Mech. Anal.* 136.2, pp. 101–117

- Existence of a radially symmetric solution when Ω is a ball.

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Characterization of minimizers

Can we find some explicit solutions?

- The problem is solved explicitely in 1D.

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Conjecture (Conca et al., Dambrine)

- When $\Omega \subset \mathbb{R}^d$ is a ball, the minimizer is also a ball:

$$B^* = B(0, r^*) = \operatorname*{argmin}_{B \in \mathcal{B}} \lambda(B)$$

Solution in a particular case

We exhibit global minimizers in low contrast regime.

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Asymptotic Expansion

- Low contrast regime: $\beta = \alpha + \varepsilon$ with $\varepsilon > 0$ small.
- Conductivity $\sigma^\varepsilon = \alpha + \varepsilon\chi_B$

Theorem (Rellich)

The first eigenvalue λ^ε of

$$-\mathrm{div}(\sigma^\varepsilon \nabla u^\varepsilon) = \lambda^\varepsilon u^\varepsilon \quad \text{in } \Omega,$$
$$u^\varepsilon = 0 \quad \text{on } \partial\Omega,$$

is an analytic function of ε in a neighbourhood of $\varepsilon = 0$ and the positive eigenfunction u^ε satisfying

$$\int_\Omega (u^\varepsilon)^2 = 1$$

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Asymptotic Expansion

We plug the following ansätze:

$$u^\varepsilon = v_0 + \varepsilon v_1 + \dots,$$
$$\lambda^\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \dots,$$

in $-\mathrm{div}(\sigma^\varepsilon \nabla u^\varepsilon) = \lambda^\varepsilon$ and $u^\varepsilon = 0$. Gather terms of similar order in ε :

$$-\mathrm{div}(\alpha\nabla v_0) = \lambda_0 v_0 \text{ in } \Omega, \quad (2.1)$$
$$v_0 = 0 \text{ on } \partial\Omega. \quad (2.2)$$

$$-\mathrm{div}(\alpha\nabla v_1) - \lambda_0 v_1 = \mathrm{div}(\chi_B \nabla v_0) + \lambda_1 v_0 \text{ in } \Omega, \quad (2.3)$$
$$v_1 = 0 \text{ on } \partial\Omega. \quad (2.4)$$

(2.3)-(2.4) has a solution if and only if (Fredholm alternative)

$$\int_\Omega \mathrm{div}(\chi_B \nabla v_0) v_0 + \lambda_1 \int_\Omega v_0^2 = 0.$$

Asymptotic Expansion

Using $\int_\Omega v_0^2 = 1$ we obtain

$$\lambda_1 = -\int_\Omega \mathrm{div}(\chi_B \nabla v_0) v_0 \implies \lambda_1 = \lambda_1(B) = \int_B |\nabla v_0|^2.$$

Theorem

If $B_\varepsilon^\star \in \mathcal{B}$ is a minimizer of $\lambda^\varepsilon(\cdot)$ then:

$$\left| \lambda_1(B_\varepsilon^\star) - \inf_{B \in \mathcal{B}} \lambda_1(B) \right| \leq C\varepsilon^{\frac{1}{2}}.$$

Optimality conditions

Theorem

- *There exists* $c^* \geq 0$ *such that whenever* B *is a measurable subset of* Ω *satisfying*

$$\{x : |\nabla v_0(x)| < c^*\} \subset B \subset \{x : |\nabla v_0(x)| \leq c^*\}$$

and $|B| = m$*, then* B *is a solution for the problem of minimizing* $\lambda_1(B)$ *over* $B \in \mathcal{B}$*.*

- *If* $\{x : |\nabla v_0(x)| = c^*\}$ *is of measure zero, then the* unique *solution (up to a set of measure zero) is the set*

$$B^* = \{x : |\nabla v_0(x)| < c^*\}.$$

This is the case if Ω *is a disk.*

The Disk Case

- $\Omega = \mathbb{B}(0,1)$ in 2D or 3D
- The solution of $-\mathrm{div}(\alpha \nabla v_0) = \lambda_0 v_0$ in Ω and $v_0 = 0$ on $\partial\Omega$ is radial: $v_0(x) = w(|x|)$

$$r^2 w_0''(r) + (d-1) r w_0'(r) + r^2 \frac{\lambda_0}{\alpha} w_0(r) = 0,$$
$$w_0'(0) = 0, \; w_0(1) = 0.$$

- In 2D, $w_0(r) = J_0(\eta_d r)$ where J_0 is the Bessel function of the first kind and η_d is its first zero.

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The Disk Case

$|\nabla v_0|^2(x) = (w_1(r))^2 := (-w_0'(r))^2$ and the solution is:

$$\lambda_1(B) = \int_B |\nabla v_0|^2 \Longrightarrow B^* = \{x : w_1(r) < c^*\}$$

where c^* is such that $|B^*| = m$.

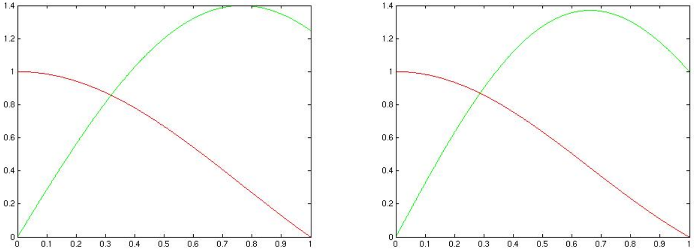

Figure : $w_0(r)$ (red), and $w_1(r) := -w_0'(r)$ (green) in dimensions $d = 2$ (left) and $d = 3$ (right), w_1 increasing on $[0, r_d^1]$ and decreasing on $[r_d^1, 1]$, and r_d^0 is such that $w_1(r_d^0) = w_1(1)$.

The Disk Case

Theorem

The solution $B^* = \min_{B\in\mathcal{B}} \lambda_1(B)$ *is of two possible types. There exists* $\overline{m} = \omega_d (r_d^0)^d$ *such that*

- *Type I: If* $m \leq \overline{m}$ *then* $B^* = B(0, (m/\omega_d)^{1/d})$ *or,*
- *Type II: If* $m > \overline{m}$ *then there exists* ξ^0 *and* ξ^1 *with* $(m/\omega_d)^{1/d} < \xi^0 < \xi^1 < 1$ *such that*

$$B^* = B(0, \xi^0) \cup \left(B(0,1) \setminus \overline{B(0,\xi^1)} \right) .$$

Theorem

When $\Omega = B(0,1)$, *for* $\beta = \alpha + \varepsilon$ *sufficiently close to* α *and* $m > \overline{m}$, $B = \mathbb{B}(0, r^*)$ *does not minimize* $\lambda^\varepsilon(B)$ *in* $\mathcal{B}$.

Low contrast regime - other geometries

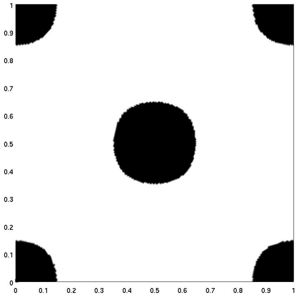

Figure : Optimal distribution of the material B (black) and A (white) when Ω is a square in low contrast regime. The set B contains the corners and the center. $m/|\Omega| \approx 14\%$.

Low contrast regime - other geometries

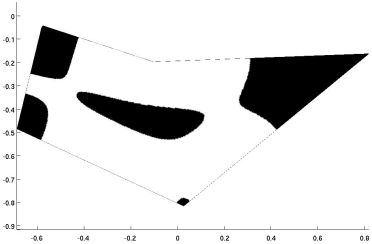

Figure : Optimal distribution of the material B (black) and A (white) when Ω is a polygon in low contrast regime. The set A contains the reentrant corner. $m/|\Omega| \approx 34\%$.

Low contrast regime - other geometries

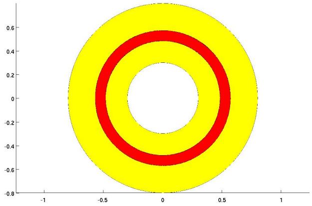

Figure : Optimal distribution of the material B (red) and A (yellow) when Ω is a ring in low contrast regime. The set B is also a ring. $m/|\Omega| \approx 17\%$.

C. Conca, A. Laurain, and R. Mahadevan (2012). "Minimization of the Ground State for Two Phase Conductors in Low Contrast Regime." In: *SIAM Journal on Applied Mathematics* 72.4, pp. 1238–1259

Global optimum in low contrast regime

- We want to prove that $B^* = \operatorname{argmin} \lambda_1(B)$ is also a minimizer of $\lambda^\varepsilon(B)$ for small ε.
- We have found minimizers of $\lambda_1(B)$ but not of $\lambda^\varepsilon(B)$, it was enough to disprove the conjecture.
- The minimizer $B_\varepsilon = \operatorname{argmin} \lambda^\varepsilon(B)$ does not necessarily converge as $\varepsilon \to 0$.
- If it does, B_ε does not necessarily converge to $B^* = \operatorname{argmin} \lambda_1(B)$.
- We need to prove first $B_\varepsilon \to B^*$ in an appropriate sense. The convergence of B_ε is linked to the convergence $\nabla u_\varepsilon \to \nabla u_0$. We need a convergence of ∇u_ε stronger than just L^2.

L^∞-convergence of the gradient

Theorem (arbitrary Ω)

For $\varepsilon > 0$ small, there exists c independent of ε and B such that

$$\|u_\varepsilon(B) - u_0\|_{H_0^1(\Omega)} \leq c\,\varepsilon^{\frac{1}{2}} \qquad \forall B \in \mathcal{B}\,.$$

Theorem (case $\Omega = \mathbb{B}(0,1)$)

Assume $\Omega = \mathbb{B}(0,1)$ and B is radially symmetric. The functions u_ε and u_0 are in $W^{1,\infty}(\Omega)$ and there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,

$$\|\nabla u_\varepsilon - \nabla u_0\|_{L^\infty(\Omega)} \leq c\sqrt{\varepsilon}.$$

Idea of the proof: the radial symmetry brings additional regularity, and use Hardy's inequality.

Quasi-optimal sets

Theorem

Let

$$r^* = (m/\omega_d)^{1/d}, \qquad \omega_d = |\mathbb{B}(0,1)|.$$

Let $B \subset \Omega$ *be a radially symmetric measurable set and* $m < \overline{m}$. *For all* $\delta > 0$, *there exists* $\varepsilon_0 = \varepsilon_0(\delta) > 0$ *and* B^*_δ *radially symmetric and containing the origin such that for all* $0 < \varepsilon \leq \varepsilon_0(\delta)$ *we have*

$$\lambda^\varepsilon \lambda^\varepsilon(B^*_\delta) \leq \lambda^\varepsilon(B) \qquad |B^*_\delta| = m,$$

and

$$\mathbb{B}(0, r^* - \delta) \subset B^*_\delta \subset \mathbb{B}(0, r^* + \delta)$$

Idea of the proof: use $\|\nabla u_\varepsilon - \nabla u_0\|_{L^\infty(\Omega)} \leq c\sqrt{\varepsilon}$ and threshold.

Global optimum in low contrast regime

- Goal: prove the existence of $\varepsilon_0 > 0$ such that

$$\lambda^{\varepsilon}(B^*) \leq \lambda^{\varepsilon}(B),$$

  for all $B \in \mathcal{B}$ and $\varepsilon \leq \varepsilon_0$, where $B^* = \mathbb{B}(0, r^*)$.
- Fact: for all $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\lambda^{\varepsilon}(B^*_{\delta(\varepsilon)}) \leq \lambda^{\varepsilon}(B)$$

  holds with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\delta(\varepsilon)$ strictly increasing.
- We need: the other inequality

$$\lambda^{\varepsilon}(B^*) \leq \lambda^{\varepsilon}(B^*_{\delta(\varepsilon)}).$$

- $B^*_{\delta(\varepsilon)}$ is "close" to B^*, otherwise no information.
- It is just enough to perform an asymptotic expansion of the eigenvalue with respect to $\delta(\varepsilon)$.

Global optimum in low contrast regime

- We prove: For all $0 < \varepsilon \leq \varepsilon_0$ and $0 < \delta \leq \delta_0$ we have

$$\lambda^\varepsilon(B^*) \leq \lambda^\varepsilon(B_\delta),$$

where B_δ is any radially symmetric set satisfying

$$\mathbb{B}(0, r^* - \delta) \subset B_\delta \subset \mathbb{B}(0, r^* + \delta).$$

- Choose $B_\delta = B^*_{\delta(\varepsilon)}$ for ε small enough

$$\lambda^\varepsilon(B^*) \leq \lambda^\varepsilon(B_\delta) = \lambda^\varepsilon(B^*_{\delta(\varepsilon)}) \leq \lambda^\varepsilon(B),$$

- Idea of the proof: find an expansion with $\rho(\delta) > 0$

$$\lambda^\varepsilon(B_\delta) = \lambda^\varepsilon(B^*) + \rho(\delta)\bar{\lambda}^\varepsilon + \mathcal{R}(\varepsilon, \delta) \text{ as } \rho(\delta) \to 0$$

and $\mathcal{R}(\varepsilon, \delta)/\rho(\delta) \to 0$ uniformly as $(\delta, \varepsilon) \to 0$. Prove then that $\bar{\lambda}^\varepsilon \geq 0$.

Global optimum in low contrast regime - type I

Theorem

If $m < \overline{m}$ there exists $\varepsilon_0 > 0$ such that for all $B \in \mathcal{B}$ we have

$$\lambda^\varepsilon(B^*) \leq \lambda^\varepsilon(B) \quad \text{for all} \ \ 0 < \varepsilon < \varepsilon_0$$

and the equality occurs only when $B = B^*$ almost everywhere in Ω.

Global optimum in low contrast regime - type II

Theorem

If $m > \overline{m}$ there exists $\varepsilon_0 > 0$ such that for all $B \in \mathcal{B}$ and for all $0 < \varepsilon < \varepsilon_0$ there exists $\xi_\varepsilon^0, \xi_\varepsilon^1$ such that

$$\lambda^\varepsilon(B_\varepsilon^*) \leq \lambda^\varepsilon(B)$$

where

$$B_\varepsilon^* = \mathbb{B}(0, \xi_\varepsilon^0) \cup \mathbb{B}(0,1) \setminus \overline{\mathbb{B}(0, \xi_\varepsilon^1)}$$

and the equality occurs only when $B = B_\varepsilon^*$ almost everywhere in Ω. In addition we have

$$(\xi_\varepsilon^0, \xi_\varepsilon^1) \to (\xi^0, \xi^1) \text{ as } \varepsilon \to 0.$$

A. Laurain. "Global minimizer of the ground state for two phase conductors in low contrast regime." In: *ESAIM Control Optim. Calc. Var.* (To appear)

Descent Algorithm-general α, β

Variational formulation for λ

$$\lambda(B) = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega \sigma(B)|\nabla u|^2}{\int_\Omega u^2} = \min_{u \in H_0^1(\Omega), ||u||_2 = 1} \int_\Omega \sigma(B)|\nabla u|^2.$$

Descent Algorithm

- Initial measurable set B_0, $|B_0| = m$.
- $\mathcal{M}(B_0, c) := |\{x : |\nabla u_{B_0}(x)| \leq c\}|$.
- $c_0 := \inf\{c : \mathcal{M}(B_0, c) \geq m\}$.
- Under suitable conditions $\mathcal{M}(B_0, c_0) = m$.
- Update $B_1 = \{x : |\nabla u_{B_0}(x)| \leq c_0\}$.

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- Update $B_1 = \{x : |\nabla u_{B_0}(x)| \leq c_0\}$.

Descent Algorithm-general α, β

Variational formulation for λ

$$\lambda(B) = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega \sigma(B)|\nabla u|^2}{\int_\Omega u^2} = \min_{u \in H_0^1(\Omega), ||u||_2 = 1} \int_\Omega \sigma(B)|\nabla u|^2.$$

Descent Algorithm

- Initial measurable set B_0, $|B_0| = m$.
- $\mathcal{M}(B_0, c) := |\{x : |\nabla u_{B_0}(x)| \leq c\}|$.
- $c_0 := \inf\{c : \mathcal{M}(B_0, c) \geq m\}$.
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Descent Algorithm-general α, β

Theorem

$\lambda(B_1) \leq \lambda(B_0)$; equality holds if and only if $B_1 = B_0$ a.e. (under extra hypotheses). If B_0 is optimal, then $B_0 = \{x : |\nabla u_{B_0}(x)| \leq c_0\}$ a.e.

Corollary

- ***The disk case.*** *$\Omega = \mathbb{B}(0, R)$, then B^* should include the origin.*
- ***The ring case.*** *The gradient of u vanishes on a circle whose center is the center of the ring. This circle is in the optimal set.*
- ***Domains with corners in two dimensions.*** *B^* contains a neighbourhood of the corners with angle smaller than π and $A^* = \Omega \setminus B^*$ contains a neighbourhood of the corners with angle greater than π.*

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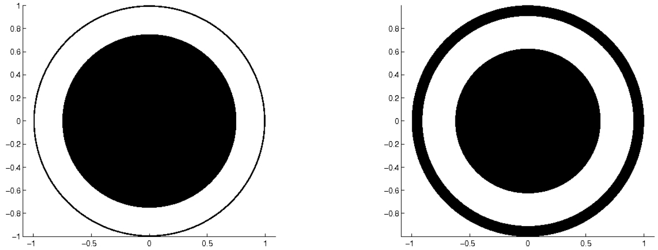

Figure : Initial domain $B_0 = \mathcal{B}(0, 0.75)$ (left). Optimal distribution (right).

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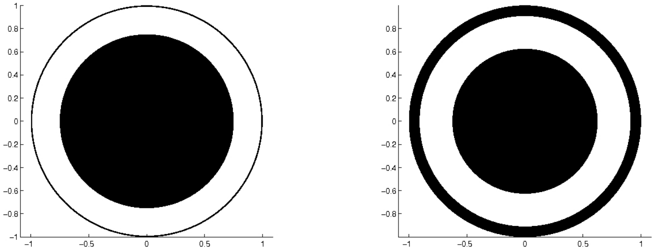

Figure : Initial domain $B_0 = \mathcal{B}(0, 0.75)$ (left). Optimal distribution (right).

Thank you for your attention !!